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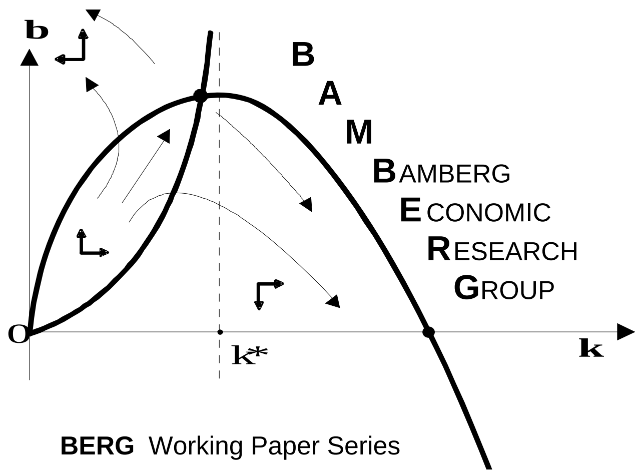

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Side effects of nonlinear profit taxes in an evolutionary market entry model: abrupt changes, coexisting attractors and hysteresis problems☆

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Abstract

In order to demonstrate that nonlinear tax systems may have surprising and potentially undesirable side effects, we develop an evolutionary market entry model in which firms decide on the basis of past profit opportunities whether or not to enter a competitive market. Our main focus is on the case of a proportional tax on positive profits. Such a piecewise-linear tax scheme induces a kink in the profit functions of firms' strategies, and may lead to abrupt changes in a market's dynamics, coexisting attractors and hysteresis problems. Since these phenomena can also be observed in more general models, a proper understanding of their basic mechanism may be helpful to explain the intricate behavior of many economic systems.

Keywords: Market entry model, replicator dynamics, evolutionary fitness, nonlinear profit taxes, stability analysis, policy implications
JEL classification: D84; E30; H20

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1. Introduction

Real tax systems often incorporate nonlinearities. As an example, consider the case of a piecewise-linear profit tax according to which the profit tax rate for firms is positive if they make a profit and zero if they make a loss. A direct consequence of such a tax system is that it causes a kink in the firms' profit function: net profits equal gross profits if firms make a loss but only amount to a fraction of gross profits if they make a profit. While there may be many reasons why policy-makers raise nonlinear profit taxes - see Slemrod (1990), Daveri and Tabellini (2000) and Mankiw et al. (2009) for reviews on optimal taxation - it is important to note that a tax-induced kink in the firms' profit function may also be of relevance for the dynamic properties of the markets involved. As is well known, the spectrum of possible behaviors of nonlinear dynamical systems clearly exceeds that of linear dynamical systems. What is presumably less known is that nonlinear dynamical systems with a kink, i.e. piecewise dynamical systems, may give rise to even richer behaviors.

In this paper we consider the effect of a nonlinear tax schedule on firms' decisions to enter a particular market. We take a behavioral approach in the sense that past realized profits are the main determinant of firms' decisions, similar to the evolutionary models of Brock and Hommes (1997), Goeree and Hommes (2000), Laselle et al. (2005), Branch and McGough (2008), Dieci and Westerhoff (2010) and Tuinstra et al. (2014). Let us illustrate the implications of a tax-induced kink in this environment from an economic perspective. Suppose that policy-makers impose a proportional tax on positive profits. The imposition of such a tax causes a kink in the market's dynamical system. As long as firms' strategies produce positive profits, firms have to pay profit taxes and the kink in the profit functions of their strategies does not matter. Put differently, the behavior of the firms then depends on the smooth curvature of their relative past profit opportunities. However, when markets start to fluctuate more strongly, for whatever reason, firms may occasionally make a loss. In such an environment, the kink in firms' relative profit functions comes into play. In particular, the marginal impact of a change in the profit generated by a strategy now depends on the level of the gross profit, which may cause a substantial change in the firms' behavior.

The goal of our paper is to show that a tax-induced kink in the firms' relative profit function may have surprising and possibly unintended side effects. In order to be able to study the consequences of nonlinear profit taxes, we develop a simple evolutionary market entry model. The structure of the model and our main results can be summarized as follows. Firms may enter a competitive commodity market. The profitability of the market is unknown to firms when they make their market entry decision; it depends to a significant extent on the number of firms that simultaneously decide to enter it. Alternatively, firms can realize a constant and positive outside profit. Moreover, firms face a piecewise-linear profit tax according to which their profit tax rate is zero if they make a loss and positive if they make a profit. Firms repeat their market entry decision at the beginning of each period and decide partly on the basis of the past net profitability of the market relative to the past net profitability of their outside option. Our model is behavioral in the sense that we assume: the higher the past relative profitability of the market, the more firms will enter it. Firms that enter the market face a downward sloped demand function and determine their supply by maximizing their expected profit. Finally, the price of the commodity adjusts such that the market clears.

As it turns out, the dynamics of our model is due to a one-dimensional nonlinear map. The map of the model is nonlinear because of the firms' market entry behavior and the piecewise-linear profit tax. The model typically has a unique interior steady state in which the number of firms that enter the market is such that the profitability of the market is equal to the profitability of the firms' outside option. As in Brock and Hommes (1997), the inner steady state of the model eventually loses its local asymptotic stability as the firms' sensitivity to past performance, the so-called 'intensity of choice', increases. To be precise, we observe a period-two cycle if firms react very strongly to profit differentials. Then either too many firms enter the market and its profitability is relatively low or too few firms enter the market and its profitability is relatively high. As in Schmitt and Westerhoff (2015), however, policy-makers can always re-establish market stability by raising the profit tax rate. The basic intuition behind this result is that higher profit tax rates reduce profit differentials, slowing down firms' market entry and exit behavior.

However, the main message of our paper is that piecewise-linear tax systems may trigger abrupt changes in the dynamics of a market, lead to coexisting attractors and cause hysteresis problems. These phenomena are a direct consequence of the tax-induced kink in the firms' profit function. As we will see, the primary bifurcation of the inner steady state of our model is a period-doubling bifurcation, i.e. we observe a low-amplitude period-two cycle once the inner steady state becomes unstable. Since fluctuations in the market are rather modest, its profitability initially remains positive. However, the amplitude of the period-two cycle increases with the firms' intensity of choice and, eventually, fluctuations in the market are so high that firms are on the edge of realizing a loss. Due to the kink in their profit function, firms sharply change their behavior in such a situation. The low-amplitude period-two cycle then suddenly stops existing, and the dynamics converges to a high-amplitude period-two cycle. In technical terms, this phenomenon is caused by a border-collision bifurcation.

Interestingly, this high-amplitude period-two cycle already exists well before the primary bifurcation, i.e. there are parameter combinations where either a locally stable steady state coexists with a locally stable high-amplitude period-two cycle or where a locally stable low-amplitude period-two cycle coexists with a locally stable high-amplitude period-two cycle. The emergence of this high-amplitude period-two cycle is also caused by the kink in the firms' profit function. Profit taxes change the difference between profits and losses so strongly that there is either a significant inflow or outflow of firms from the market. Speaking again in technical terms, the high-amplitude period-two cycle is created via a saddle-node bifurcation of a period-two cycle (in addition to the stable cycle, another unstable period-two cycle is born at the bifurcation).

Since the map of the model is one-dimensional, we are able to compute (and visualize) the full basins of attraction of the coexisting attractors. Although the basins of attraction of the coexisting attractors are well pronounced, occasional exogenous shocks can push the dynamics from one basin of attraction to another. The dynamics is then characterized by periods of low volatility alternating with periods of high volatility. Due to the mixture of exogenous shocks, transient dynamics and attractor switching, the overall behavior of the market is quite erratic.

The bifurcation structure of the model may furthermore give rise to remarkable hysteresis effects. For instance, if the market dynamics is characterized by a low-amplitude period-two cycle which coexists with a high-amplitude period-two cycle, a reduction in the profit tax rate may destroy the low-amplitude period-two cycle and push the dynamics to the high-amplitude period-two cycle. A return to the previous profit tax rate may not suffice to drive the dynamics back to the low-amplitude period-two cycle since the system may remain in the basin of attraction of the coexisting high-amplitude period-two cycle. A temporary increase to a much higher profit tax rate may then be needed to drive the market back towards the low-amplitude period-two cycle. However, hysteresis effects may also be of relevance for policy-makers' tax revenues. Small changes in the profit tax rate may cause substantial jumps in tax revenues, implying that policy-makers' tax revenue function is discontinuous. Due to the coexistence of attractors, a return to the previous profit tax rate does not necessarily ensure that policy-makers are able to realize the previous tax revenue.

In order to make our analysis as clear as possible, we keep our evolutionary market entry model as simple as possible. Nevertheless, a detailed robustness analysis reveals that similar effects of nonlinear tax systems can also occur in much more general models. For instance, we also find these effects in models in which consumer demand is isoelastic instead of linear; in which firms' market entry decisions are modeled via a discrete choice approach instead of the exponential replicator dynamics approach we use here; in which firms have naive expectations instead of rational expectations about market prices; in which firms switch between naive and rational price expectations instead of having a single prediction rule; in which firms' market entry decisions depend on a moving average of the past profits of the market instead of the last observed profit; or in which the outside profits of firms evolve endogenously instead of being constant.

Note that there are many other economic settings in which market participants rely on heterogeneous strategies and switch between them with respect to their past performance. For instance, in the asset-pricing model of Brock and Hommes (1998), market participants have the choice between technical and fundamental trading rules; they prefer those rules that produced higher profits in the last trading periods. De Grauwe and Grimaldi (2006) develop a similar evolutionary approach to explain the volatile behavior of foreign exchange markets. Hommes and Zeppini (2014) suggest a behavioral model of technological change in which firms' decisions to innovate or imitate is profit-dependent. Related to this, Zeppini (2015) explores consequences of a pollution tax in a model in which firms' selection between a clean and a dirty technology depends on the profitability of these technologies and on network externalities. Finally, we mention Neugart and Tuinstra (2003), who study fluctuations in the demand for higher education in a model in which students' choices depend on past wage differentials. Hommes (2013) provides many more examples and empirical support for this line of research. Our analysis suggests that regulating these markets with respect to profit taxes, income taxes or wealth taxes will have an impact on their dynamics. In particular, if these markets are subject to piecewise-linear taxes, the fitness functions of agents' strategies may be kinked, leading to the emergence of surprising dynamic phenomena. Note that we are not arguing against the imposition of nonlinear taxes. Our goal is merely to point out that nonlinear taxes may have unexpected implications for the dynamic behavior of economic systems, an aspect that policy-makers may wish to bear in mind.

Before moving on to our analysis, a few additional remarks are in order. In the last couple of years, a number of papers have appeared that seek to explain various economic phenomena on the basis of piecewise maps. To name but a few, Day (1982), Day and Shaffer (1987), Hommes (1995), Matsuyama (1999), Matsuyama (2007), Gardini et al. (2008), Kubin and Gardini (2013) and

Sushko et al. (2014) develop piecewise models to study irregular fluctuations in economic activity; Huang and Day (1993), Huang et al. (2010), Huang and Zheng (2012) and Tramontana et al. (2010, 2013) generate piecewise maps to explore the dynamics of financial markets; and Agliari et al. (2011) and Commendatore et al. (2014, 2015) propose using piecewise models to investigate the process of industrial agglomeration. The success of these contributions is based at least in part on the fact that piecewise maps may give rise to nonstandard bifurcations. These include, for instance, abrupt and significant changes in the dynamics of a model or the appearance/disappearance of coexisting attractors as a model parameter varies. Of course, knowledge of such bifurcations may be important in improving our understanding of how economic systems function, as illustrated by the aforementioned papers. For a general introduction to this research area and up-to-date techniques for exploring piecewise maps, see Avrutin et al. (2016).

The rest of our paper is organized as follows. In Section 2, we present our model and derive a number of preliminary analytical results. In Section 3, we discuss a number of surprising effects of nonlinear tax systems. In Section 4, we check the robustness of our results. In Section 5, we summarize our main findings and discuss various avenues for future research.

2. A stylized evolutionary market entry model

In this section, we introduce a simple evolutionary market entry model to study the effects of nonlinear tax systems. After introducing the market environment and presenting the dynamical system governing market entry decisions in Section 2.1, we analyze its steady states and their dynamic properties in Section 2.2. To make our results as clear as possible, we use a stylized benchmark model in this section. In Section 4, we will demonstrate that the qualitative results remain valid when extending the model.

2.1. The market environment and market entry decisions

In our model, firms can choose to enter a competitive market for a homogeneous commodity. The profitability of competing in this market is unknown to these firms when making their market entry decision; it depends to a substantial extent on the number of other firms that decide to enter this market. If firms abstain from entering the market, they will realize constant (and positive) outside profits. Either way, firms have to pay taxes on positive profits. Each firm's market entry decision is repeated at the beginning of each period, and depends on the past profitability of the market relative to the profitability of their outside option. The higher the relative past profitability of the market, the more firms will enter the market. Firms that enter the market engage in market research, enabling them to determine their profit-maximizing supply. In the inner steady state of the model, the number of active firms in the market is such that the profitability of entering the market equals the profitability of the outside option. However, this steady state is not necessarily stable, and there may be other (coexisting) types of attractors.

Let us now turn to the details of our model. Let D_t and S_t represent consumer demand and supply by firms in the market at time step t, respectively. The market clears in every period, that is

$$D_t = S_t. \tag{1}$$

Consumer demand depends negatively on the current commodity price p_t, and is formalized as

$$D_t = 1 - p_t. \tag{2}$$

We consider a fixed number N of potential producers, which have identical cost functions for producing the commodity. By normalizing the total number of firms to $N = 1$, the total supply generated by firms can be expressed as

$$S_t = n_t q_t, \tag{3}$$

where n_t stands for the fraction of firms entering the market and q_t signifies the supply generated by a single firm.

Firms face a quadratic cost function $C_t = 0.5q_t^2 + F$ with fixed costs $F > 0$. In addition, firms may need to pay profit taxes, where $0 \leq \tau < 1$ describes the tax rate levied on positive profits. Firms' optimal production decisions are derived from expected profit maximization, i.e.

$$\underset{q_t}{argmax}\ \pi_t^e = \underset{q_t}{argmax} \begin{cases} (1-\tau)(p_t^e q_t - C_t) & for \quad p_t^e q_t - C_t \geq 0 \\ p_t^e q_t - C_t & for \quad p_t^e q_t - C_t < 0 \end{cases} \tag{4}$$

and is given by $q_t = p_t^e$, where p_t^e denotes the firms' price expectations. We assume that firms have rational expectations about market clearing prices, i.e.

$$p_t^e = p_t \tag{5}$$

and thus the optimal supply generated by a single active firm amounts to

$$q_t = p_t. \tag{6}$$

In Brock and Hommes (1997), rational expectations are available at constant information costs. Here we may simply assume that rational expectations are free.

Before we continue with our model, note that by combining (1), (2), (3) and (6), we obtain

$$p_t = \frac{1}{1+n_t}. \tag{7}$$

Accordingly, the market clearing price in period t depends negatively on the current number of firms active in this market. Of course, relation (7) also holds in a steady state, implying that $p^* = \frac{1}{1+n^*}$ and $n^* = \frac{1-p^*}{p^*}$. Since the total number of firms is normalized to 1, it follows that $0 \leq n_t \leq 1$, $0.5 \leq p_t \leq 1$ and $0 \leq D_t = S_t \leq 0.5$.

The fraction of firms entering the market is updated over time according to an evolutionary approach, which is based on the idea that firms tend to choose the alternative that was most profitable in the previous period. Although firms have only an incomplete understanding of their economic environment, they learn about the profitability of their alternatives from their own past experience and by observing the success of other firms.[1] Since firms must only pay taxes on positive profits, net profits realized by firms active in the market can be expressed as

$$\pi_t = \begin{cases} (1-\tau)(0.5p_t^2 - F) & if \quad p_t \geq \sqrt{2F} \\ 0.5p_t^2 - F & if \quad p_t < \sqrt{2F} \end{cases}. \tag{8}$$

Note that for $p_t = \sqrt{2F}$ we obtain $\pi_t = 0$. Exactly at this price level, the profit function (8) has a kink for any positive profit tax rate.[2] Net profits realized by firms that are not active in the market are equal to $\hat{\pi} = (1-\tau)\tilde{\pi} > 0$. To justify this simplifying assumption, we may regard the market under consideration as a small local market for which the firms' market entry and exit behavior is of relevance, while the outside option stands for a large global market that is insensitive to the inflow and outflow of a limited number of additional firms. Alternatively, we may assume that firms can earn risk-free profits $\tilde{\pi}$ by investing in safe capital markets.

Following, for instance, Tuinstra et al. (2014), we model the number of active firms in the market via the exponential replicator dynamics. Hence,

$$n_t = \frac{n_{t-1}exp[\beta\pi_{t-1}]}{n_{t-1}exp[\beta\pi_{t-1}] + (1-n_{t-1})exp[\beta\hat{\pi}]} = \frac{n_{t-1}}{n_{t-1} + (1-n_{t-1})exp[\beta(\hat{\pi} - \pi_{t-1})]}. \tag{9}$$

Obviously, the higher the market's past profitability, the more firms will enter it. Parameter $\beta > 0$ is the intensity of choice, and measures how sensitive firms are in selecting the most profitable alternative. The higher β is, the more firms will choose the alternative that yielded higher profits in the previous period. Besides

[1] Note that, although firms base their entry decision on past performance, we assume that once they have entered the market, they learn about the number of competitors and have rational expectations about the market clearing price. Alternatively, we could make the assumption that they also have naive (backward-looking) expectations about the market clearing price. The analysis in Section 4 shows that this alternative assumption leads to qualitatively similar results.

[2] Our results are robust with respect to the introduction of a tax break, i.e. when we let the first $\pi_0 > 0$ units of profits be exempted from the profit tax. The kink in the profit function will then be at π_0 instead of at 0, but qualitatively the results will be similar.

directly observing the performance differential between the commodity market and their outside option, (9) also captures the idea that firms tend to imitate the behavior of more successful firms. Indeed, as can be seen from equation (9), due to imitation, a very high (low) fraction of active firms in period t tends to lead to a high (low) fraction of active firms in the subsequent period, even if market profits are below (above) payoffs from the outside option. Imitation therefore implies that profit differentials will have a significant effect on the number of active firms for several subsequent periods, which introduces a modest level of inertia in the evolutionary dynamics, not present in the standard discrete choice model used by, e.g. Brock and Hommes (1997, 1998). The origins of the (exponential) replicator dynamics can be traced back to Hofbauer and Sigmund (1988) and Hofbauer and Weibull (1996), and related economic applications include, amongst others, Droste et al. (2002), Branch and McGough (2008) and Dindo and Tuinstra (2011).

2.2. Steady states and stability

Combining (7)-(9) reveals that the number of active firms evolves according to

$$n_t = f(n_{t-1}) = \begin{cases} \frac{n_{t-1}}{n_{t-1}+(1-n_{t-1})exp[\beta(1-\tau)(\tilde{\pi}-\frac{0.5}{(1+n_{t-1})^2}+F)]} & if \quad n_{t-1} \leq \frac{1-\sqrt{2F}}{\sqrt{2F}} \\ \frac{n_{t-1}}{n_{t-1}+(1-n_{t-1})exp[\beta((1-\tau)\tilde{\pi}-\frac{0.5}{(1+n_{t-1})^2}+F)]} & if \quad n_{t-1} > \frac{1-\sqrt{2F}}{\sqrt{2F}} \end{cases}. \tag{10}$$

Note that once n_t is determined, p_t can be derived via (7), and π_t follows from (8). The dynamic properties of our model therefore depend solely on the one-dimensional nonlinear map (10).

Inserting $n^* = n_t = n_{t-1}$ in (10) and solving for n^* reveals that our model may have up to three steady states, namely $n_1^* = 0$, $n_2^* = \frac{1-\sqrt{2(F+\tilde{\pi})}}{\sqrt{2(F+\tilde{\pi})}}$ and $n_3^* = 1$. The two boundary steady states n_1^* and n_3^* always exist, while the existence of the inner steady state n_2^* obviously requires that $0 < n_2^* < 1$. To ensure that $n_2^* > 0$, we assume from now on that A1: $\tilde{\pi}+F < 0.5$. Economically, assumption A1 implies that profits in the market exceed payoffs from the outside option at the first steady state and thus motivates firms to enter the market when no other firms do. In addition, assuming that $\tilde{\pi} + F > 0.125$ would already suffice to guarantee that $n_2^* < 1$. In fact, for $\tilde{\pi}+F = 0.125$, market profits would be equal to outside profits at the third steady state. However, in order to be able to fully explore the dynamic effects of nonlinear profit taxes, we assume that profits at $n_3^* = 1$ are negative. We thus impose the stronger restriction A2: $F > 0.125$. Together, A1 and A2 imply that $0.125 < F < 0.5$ and $0 < \tilde{\pi} < 0.375$.

Moreover, let n_k be the number of active firms for which profits from the market become zero. Given restrictions A1 and A2, the three steady states of map (10) have the following properties: $n_1^* = 0 < n_2^* = \frac{1-\sqrt{2(F+\tilde{\pi})}}{\sqrt{2(F+\tilde{\pi})}} < n_k = \frac{1-\sqrt{2F}}{\sqrt{2F}} < n_3^* = 1$. The corresponding steady-state prices can be ordered as $p_3^* = 0.5 < p_k = \sqrt{2F} < p_2^* = \sqrt{2(F+\tilde{\pi})} < p_1^* = 1$; the ranking for the steady-state net profits is $\pi_3^* = 0.125 - F < \pi_k = 0 < \pi_2^* = (1-\tau)\tilde{\pi} < \pi_1^* = (1-\tau)(0.5-F)$. Note that the inner steady state of the model does not depend on β or τ. Hence, a change in the firms' intensity of choice or a change in the profit tax rate has no impact on the inner-steady state number of firms or on the inner steady-state market clearing price. In contrast, both an increase in fixed costs F and an increase in the profitability of the firms' outside option $\tilde{\pi}$ make the market less attractive for firms, driving up its inner steady-state equilibrium price. The position of the kink in the map depends only on F. The higher the fixed costs, the lower the number of firms that are able to make a profit in the market.

A sufficient condition ensuring that a steady state of a one-dimensional nonlinear map is locally asymptotically stable is given by $|f'(n^*)| < 1$. Since $f'(n_1^*) = exp[\beta(1-\tau)(0.5-(F+\tilde{\pi}))]$ and A1: $\tilde{\pi} + F < 0.5$, it follows that $f'(n_1^*) > 1$. As a result, $n_1^* = 0$ is an unstable steady state. The economic reason for this outcome is that our setup implies that a small number of firms realizing comparatively high profits will attract additional firms to the market. Furthermore, the first derivative of map (10) evaluated at $n_3^* = 1$ is $f'(n_3^*) = exp[\beta(\tilde{\pi}(1-\tau) + F - 0.125]$. Since A2: $F > 0.125$, $f'(n_3^*) > 1$ and thus $n_3^* = 1$ is also an unstable steady state. The explanation for this outcome is a mirror image to that for the first steady state. A high number of active firms realizing comparatively low profits drives some of these firms towards the outside option.

For the inner steady state, we obtain $f'(n_2^*) = 1-2\beta(1-\tau)p_2^*(1-p_2^*)(p_2^*-0.5)$. Since $0.5 < p_2^* < 1$, it is always guaranteed that $f'(n_2^*) < 1$. However, $f'(n_2^*) = -1$ at $\beta(1-\tau) = (p_2^*(1-p_2^*)(p_2^*-0.5))^{-1}$. Hence, local asymptotic stability of n_2^* requires that

$$\beta(1-\tau) < \frac{1}{(\sqrt{2(F+\tilde{\pi})})(1-\sqrt{2(F+\tilde{\pi})})(\sqrt{2(F+\tilde{\pi})}-0.5)}. \tag{11}$$

Stability condition (11) has a number of important policy implications. As in Brock and Hommes (1997), an increase in the intensity of choice eventually renders the inner steady state unstable. If firms have a growing tendency to switch to the more profitable strategy, the market eventually becomes unstable since aggregate demand fluctuates too strongly. As in Schmitt and Westerhoff (2015), however, an appropriate increase in the profit tax rate can always re-establish market stability. The higher the profit tax rate, the lower the profit differential between firms' strategies. This slows down firms' market entry and exit behavior. Recall that in Brock and Hommes (1997) and in Schmitt and Westerhoff (2015), firms are always active in the same market and switch between naive and rational expectations, while in our setup, firms always use rational price expectations and switch between being active on the market and their outside option. It is interesting to see that profit taxes have a stabilizing impact in both model environments which are, of course, related but also differ in various ways.

It can furthermore be shown that the critical bifurcation value on the right-hand side of (11) decreases with F and $\tilde{\pi}$ up to $F = \frac{2+\sqrt{3}}{12-\tilde{\pi}}$ and $\tilde{\pi} = \frac{2+\sqrt{3}}{12-F}$, respectively, and then increases again. From this perspective, the impact of fixed costs and outside profits on market stability may be regarded as ambiguous. Finally, there is strong evidence that the primary bifurcation of the inner steady state is a period-doubling bifurcation, i.e. the inner steady state becomes unstable and a locally stable period-two cycle emerges. Indeed, a necessary condition for the emergence of a period-doubling bifurcation is that the slope of the map becomes steeper than -1 at the steady state. Combined with supporting numerical evidence (see, in particular, Figures 5 and 6), this is usually regarded as strong evidence of a period-doubling bifurcation. Unfortunately, the period-two cycle cannot be expressed analytically and thus its local stability can only be studied numerically. See Gandolfo (2008) for an introduction to the theory of nonlinear dynamical systems.

We can summarize our main analytical results in the following proposition.

Proposition 1: Given assumptions A1 *and* A2*, map (10) has three steady states. The boundary steady states* $n_1^* = 0$ *and* $n_3^* = 1$ *are always unstable while the inner steady state* $n_2^* = (1 - \sqrt{2(F + \tilde{\pi})})/\sqrt{2(F + \tilde{\pi})}$ *is locally asymptotically stable for* $\beta(1 - \tau) < (\sqrt{2(F + \tilde{\pi})}(1 - \sqrt{2(F + \tilde{\pi})})(\sqrt{2(F + \tilde{\pi})} - 0.5))^{-1}$*. Moreover, the loss of stability is due to a period-doubling bifurcation.*

Proposition 1 offers valuable insights into important properties of our model. However, our stability analysis only holds locally, and there may be other types of attractors. Also note that, although the profit tax rate has an effect on the local stability properties of the inner steady state, these local stability properties are unaffected by the kink in the profit function. This is due to the fact that at the inner steady state, market profits are always strictly positive. However, as we will see in the next section, the piecewise-linear character of the profit function plays an important role for the model's global dynamics.

3. Dynamic effects of nonlinear tax systems

We show that piecewise-linear profit taxes may help to stabilize the fluctuations of a market, but may also create abrupt changes in its dynamics, generate coexisting attractors and cause hysteresis problems. The erratic behavior of many economic systems may therefore be explained, in part, by the nonlinearity of their underlying tax schemes. As a base parameter setting for our simulations, we use $F = 0.26$, $\tilde{\pi} = 0.05$, $\beta = 42$ and $\tau = 0.5$. This parameter setting allows us to create a good visualization of our results, and satisfies our economic assumptions. We continue as follows. In Section 3.1, we illustrate some properties of our model for our base parameter setting. In Section 3.2, we explore how the four parameters of our model affect its dynamic behavior. In Section 3.3, we highlight a number of economic effects of nonlinear profit taxes. In Section 3.4, we demonstrate that the appearance and disappearance of coexisting attractors is due to a sequence of a saddle-node bifurcation of a period-two cycle, a period-doubling bifurcation and a border-collision bifurcation.

3.1. An illustrative example

Let us start by summarizing a number of properties of our model for our base parameter setting. The top left panel of Figure 1 shows how firms' net profit depend, in general, on their gross profit. While the slope of this function is $+1$ if firms make a loss, it reduces to $1-\tau=0.5$ if they make a profit. Obviously, the firms' profit function has a kink where profit is zero. The top right panel of Figure 1 documents how the net profits realized depend on the number of active firms in the market. At the first steady state we have that $n_1^*=0$, $p_1^*=1$ and $\pi_1^*=0.12$; at the third steady state we have that $n_3^*=1$, $p_3^*=0.5$ and $\pi_3^*=-0.135$. The inner steady state is located at the intersection point of the profit function of the market (solid line) and the profit function of the outside option (dashed line), yielding $n_2^*=0.27$, $p_2^*=0.787$ and $\pi_2^*=(1-\tau)\tilde{\pi}=0.025$.[3] Obviously, profitability decreases with the number of firms active in the market. Since profit taxes must only be paid on positive profits, the profit function for the market has a kink at $\pi_k=0$, corresponding to $n_k=0.387$ and $p_k=0.721$.

The solid black line in the bottom left panel of Figure 1 shows the shape of map (10). The three steady states in our model are located at the point

[3] All three steady states in our model can be derived directly from the exponential replicator dynamics (9). For $n_{t-1}=0$, $\pi_{t-1}=\hat{\pi}$ and $n_{t-1}=1$, the number of active firms comes at rest, i.e. $n_t=n_{t-1}$. In particular, the inner steady state has the economically desirable property that the number of active firms in the market is such that the profitability of being active in the market is equal to the profitability of the firms' outside option. As discussed in Section 4, this is not the case if we apply the discrete choice approach of Brock and Hommes (1997, 1998).

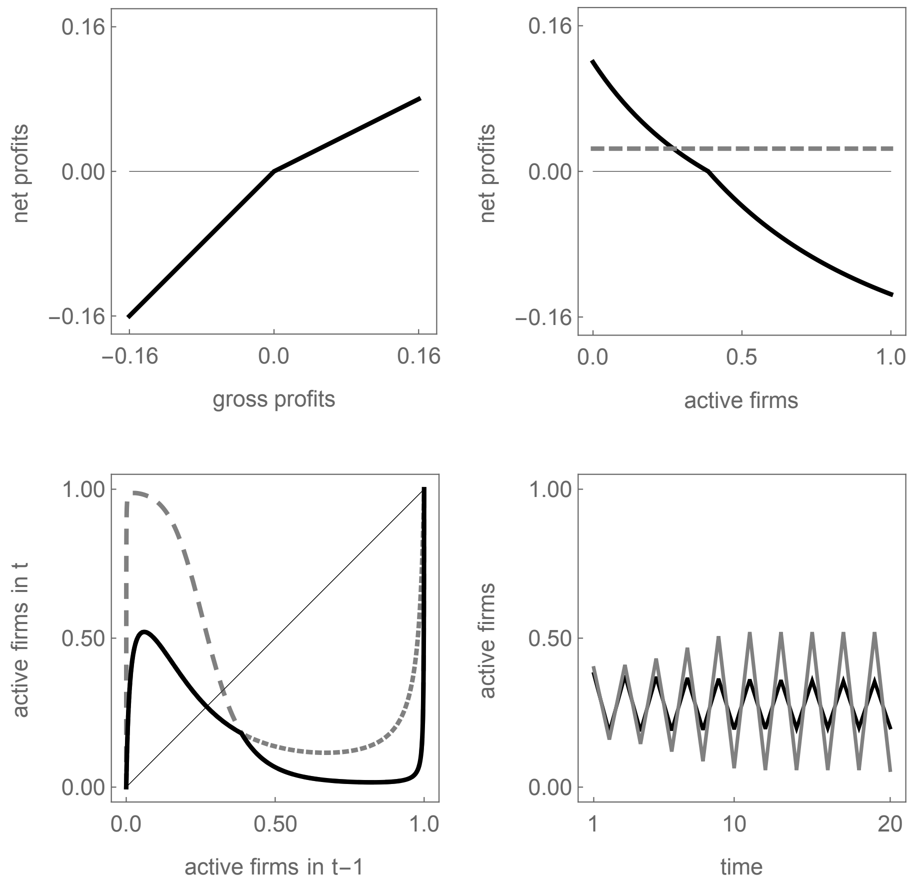

Figure 1: The top left panel shows firms' net profits versus their gross profits. The top right panel shows realized net profits in the market (solid line) and realized net outside profits (dashed line) versus the number of active firms in the market. The solid, dotted and dashed lines in the bottom left panel show map (10), the upper branch of map (10) and the lower branch of map (10), respectively. The bottom right panel shows times series for initial conditions $n_0 = 0.38$ (black line) and $n_0 = 0.40$ (gray line). Base parameter setting: $F = 0.26$, $\tilde{\pi} = 0.05$, $\tau = 0.5$ and $\beta = 42$.

where map (10) crosses the 45-degree line. Since the absolute value of the slope of map (10) is larger than one at all three steady states, none of the model's steady states is stable. In fact, the critical values of the intensity of choice and of the profit tax rate for the inner steady state to be stable are $\beta_c = 41.57$ (given

$\tau = 0.5$) and $\tau_c = 0.505$ (given $\beta_c = 42$), respectively. Map (10) is nonlinear because of the firms' market entry behavior and the profit tax-induced kink. The dotted gray line depicts the upper branch of map (10), coinciding with map (10) for $n_t \leq n_k$ (where profits, and therefore the profit tax, are positive); the dashed gray line shows the lower branch of map (10), coinciding with map (10) for $n_t > n_k$ (where the profit tax is zero, due to negative profits). Of course, neither the upper nor the lower branch of map (10) has a kink. Finally, the bottom right panel of Figure 1 presents time series of map (10) for our base parameter set. For the initial condition $n_0 = 0.38$, the dynamics converges towards a low-amplitude period-two cycle (black line); for the initial condition $n_0 = 0.40$, the dynamics converges towards a high-amplitude period-two cycle (gray line).

3.2. The role of the model's four parameters

After presenting the properties of the model for our base parameter setting, we now broaden our view and discuss how the behavior of the model depends on its four parameters. In particular, we study how the map of the model and its dynamics react to parameter changes. Let us continue with Figure 2. In all four panels of this figure, the solid line represents the map for our base parameter setting. The top left panel illustrates the dependence of map (10) on the firms' intensity of choice β. If the number of active firms is very low in period t, it will remain rather low in period $t + 1$. This is due to firms' imitation behavior, as expressed by the exponential replicator dynamics (9). However, if the number of firms active in period t starts to increase, then the number of firms active in period $t + 1$ also increases. Indeed, as long as there are not too many active firms, firms will realize substantial profits. The precise number of firms which enter the market depends on the intensity of choice. For instance, for $\beta = 42$ (solid line), the number of firms active in the market increases from $n_t = 0.1$ to about $n_{t+1} = 0.49$, while for $\beta = 63$ (dotted line) and $\beta = 21$ (dashed line), the respective numbers are roughly $n_{t+1} = 0.74$ and $n_{t+1} = 0.25$. As the number of firms active in period t increases further, the market's profitability decreases and thus the number of firms active in period $t+1$ also decreases. At the inner steady state, market profit is identical to the payoff from the firms' outside option; at the kink, market profit has dropped to zero. Note that the intensity of choice does not affect the location of the inner steady state or the location of the kink in the map. If the number of active firms exceeds $n_k = 0.387$, active firms suffer losses. The higher the intensity of choice, the fewer firms will therefore enter the market. If n_t is very high, the firms' imitation behavior becomes dominant, implying that n_{t+1} will also be high.

The top right panel of Figure 2 depicts map (10) for different values of the profit tax rate (dotted line: $\tau = 0.25$, solid line: $\tau = 0.5$, dashed line: $\tau = 0.75$). Note that the three curves are identical to those in the top left panel for $n_t < n_k$. Why is this the case? The upper branch of map (10) depends on the expression $\beta(1-\tau)$. For the dotted line we have in the top left panel $\beta(1-\tau) = 63(1-0.5) = 31.5$ and in the top right panel $\beta(1-\tau) = 42(1-0.25) = 31.5$; for the solid line we obviously have in both top panels $\beta(1-\tau) = 42(1-0.5) = 21$; for the dashed line we have in the top left panel $\beta(1-\tau) = 21(1-0.5) = 10.5$ and in the top right panel $\beta(1-\tau) = 42(1-0.75) = 10.5$. Hence, for $n_t < n_k$, the three lines in the top left and top right panel are identical. From an economic perspective, this implies that, as long as the model dynamics is restricted to the upper branch of map (10), any undesirable change in the firms' intensity of choice can, at least in principle, be offset exactly by an appropriate adjustment in the profit tax rate such that the dynamics, say a low-amplitude period-two cycle around the inner steady state of the model, remains as it was. For instance, if the intensity of choice increases from $\beta = 21$ to $\beta = 42$, an increase in the profit tax rate from $\tau = 0.5$ to $\tau = 0.75$ preserves the shape of map (10) to the left of n_k. To the right of n_k, this relation does not hold. Scrutiny of map (10) immediately reveals that the multiplicative nature of β and $(1-\tau)$ exists only in its upper branch. However, we still observe that the number of firms active in period $t+1$ increases with the profit tax rate. This is because higher profit tax rates negatively affect the profitability of the firm's outside option, making entering the market relatively more attractive for firms. Note also that the kink

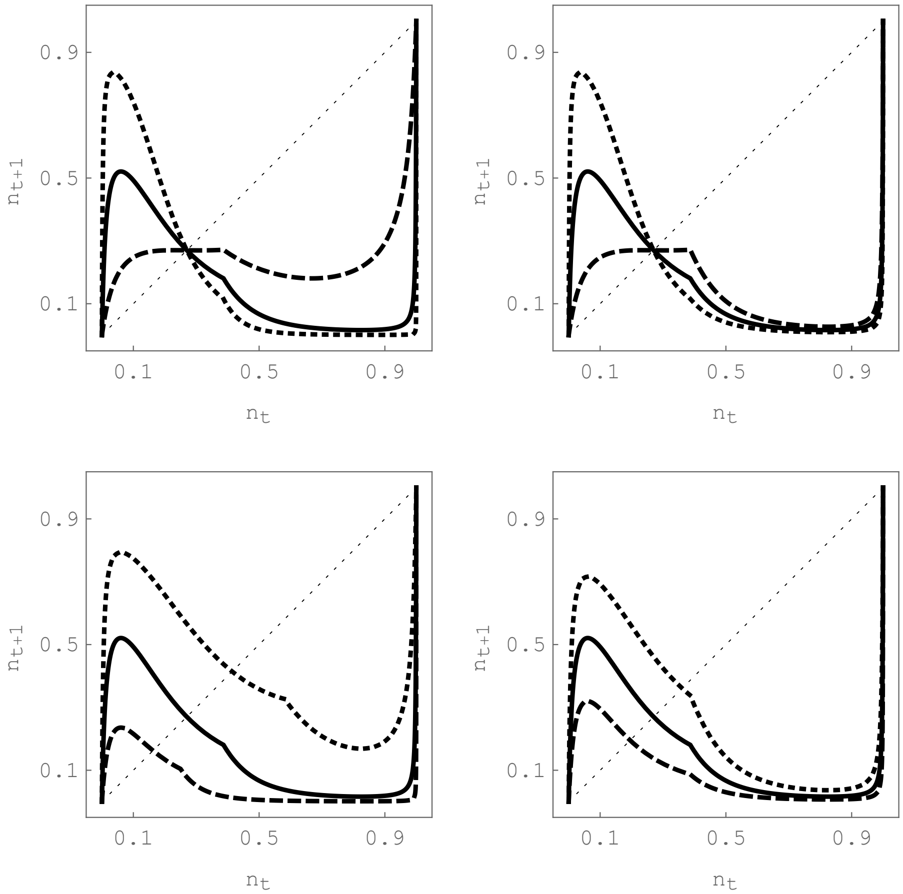

Figure 2: The shape of map (10) for different parameter constellations. All panels: the solid line shows the shape of map (10) for our base parameter setting $F = 0.26$, $\tilde{\pi} = 0.05$, $\tau = 0.5$ and $\beta = 42$. Top left panel: the dotted line emerges for $\beta = 63$ and the dashed line for $\beta = 21$. Top right panel: the dotted line emerges for $\tau = 0.25$ and the dashed line for $\tau = 0.75$. Bottom left panel: the dotted line emerges for $F = 0.2$ and the dashed line for $F = 0.32$. Bottom right panel: the dotted line emerges for $\tilde{\pi} = 0.01$ and the dashed line for $\tilde{\pi} = 0.09$.

becomes more pronounced for high tax rates (see, e.g. the path of the dashed line).

The bottom left panel of Figure 2 represents the effects of fixed costs. Here the dotted line emerges for $F = 0.2$; the solid line for $F = 0.26$; and the dashed line for $F = 0.32$. The lower the firms' fixed costs, the more market profits they can make and thus the more firms will enter the market. As can be seen, both the inner steady state and the kink increase as fixed costs decrease. In contrast, a reduction in the firms' outside profit opportunities (see lower right panel of Figure 2; dotted line: $\tilde{\pi} = 0.01$, solid line: $\tilde{\pi} = 0.05$, dashed line: $\tilde{\pi} = 0.09$) does not affect the kink in map (10). However, the lower the profitability of the firms' outside option, the more likely it is that they will enter the market, and the more active firms will be at the inner steady state.

Based on these insights, we will now discuss how the four parameters of the model may affect its dynamics. For this purpose, we make use of bifurcation diagrams, as depicted in Figure 3. In the top left panel of Figure 3, we increase parameter β in 400 discrete steps from 19 to 65. Recall that higher values of the intensity of choice imply that firms react stronger to profit differentials of their strategies. As predicted by our analytical results, the dynamics of our model converges for $\beta < 41.57$ and for initial conditions close to the inner steady state towards $n_2^* = 0.27$, followed by the emergence of a low-amplitude period-two cycle. At about $\beta = 42.62$, the low-amplitude period-two cycle vanishes abruptly, and we observe a convergence towards a high-amplitude period-two cycle. However, for initial conditions further away from the inner steady state, this high-amplitude period-two cycle already emerges at about $\beta = 39.63$, i.e. well before the primary bifurcation. Moreover, the high-amplitude period-two cycle turns, via a cascade of period-doubling bifurcations, into chaotic dynamics as the intensity of choice increases further.

How does the profit tax rate affect the model dynamics? The top right panel of Figure 3 shows a bifurcation diagram of map (10) for our base parameter setting in which we vary parameter τ in 400 discrete steps between 0 and 1. The general picture is that an increase in the profit tax rate manages to stabilize the dynamics. However, we observe again the same qualitative bifurcation structure as with respect to an increase in the firms' intensity of choice, albeit in reverse order. In the absence of profit taxes, the dynamics converges towards a high-amplitude period-two cycle and the amplitude of this cycle decreases with the profit tax rate. Between $\tau = 0.4925$ and $\tau = 0.5051$, the high-amplitude period-

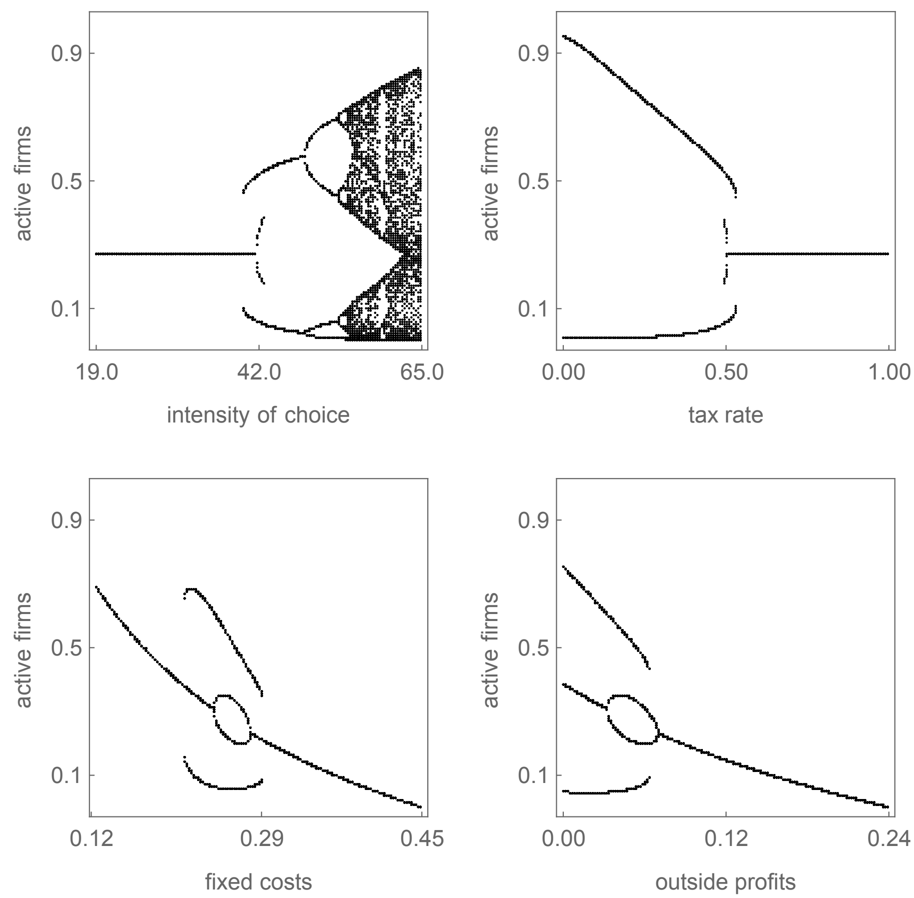

Figure 3: The top left, top right, bottom left and bottom right panels show bifurcation diagrams of map (10) for the intensity of choice, the profit tax rate, the fixed costs and the outside profits, respectively. The bifurcation parameters are increased in 400 discrete steps as indicated on the axes. Simulations are repeated for different initial conditions. Base parameter setting: $F = 0.26$, $\tilde{\pi} = 0.05$, $\tau = 0.5$ and $\beta = 42$.

two cycle coexists with a low-amplitude period-two cycle; between $\tau = 0.5051$ and $\tau = 0.5325$, it coexists with the inner steady state of the model. For $\tau > 0.5325$, the dynamics eventually approaches $n_2^* = 0.27$ for almost all initial conditions. So far, we can conclude that the numerical investigation conducted in this section confirms the analytical results obtained in the previous section: an increase in the intensity of choice has a destabilizing effect on the model's dynamics, and this destabilizing effect may be countered by an appropriate increase in the profit tax rate.

The bottom left panel of Figure 3 reveals how fixed costs affect the dynamics of our model. In this bifurcation diagram, parameter F is varied within the boundaries given by assumptions A1 and A2, i.e. between $F = 0.125$ and $F = 0.45$. As predicted by our analytical results, the inner steady state of the model decreases with F and is locally stable for $F < 0.2422$ and $F > 0.2799$. Within the range $0.2422 < F < 0.2799$, the dynamics converges towards a low-amplitude period-two cycle, but only if initial conditions are near the inner steady state. For other initial conditions, the dynamics may settle between $F = 0.2120$ and $F = 0.2919$ on a high-amplitude period-two cycle.

Similar results are observed for parameter $\tilde{\pi}$, as depicted in the bottom right bifurcation diagram of Figure 3. Taking into account assumptions A1 and A2, we increase outside profits from 0 to 0.24. The higher the outside profits, the less attractive it is to enter the market. Consequently, n_2^* decreases with $\tilde{\pi}$. Between $0.0322 < \tilde{\pi} < 0.0699$, the inner steady state of the model is not attracting. Note that the ambiguous local stability effects of F and $\tilde{\pi}$ with respect to n_2^* can be related to stability condition (11). As discussed in Section 2, the right-hand side of (11) reaches a minimum for $F = \frac{2+\sqrt{3}}{12} - \tilde{\pi} = 0.2610$ or $\tilde{\pi} = \frac{2+\sqrt{3}}{12} - F = 0.0510$, respectively. For these values, which are close to our base parameter setting, stability condition (11) is violated. For higher or lower values of F or $\tilde{\pi}$, however, stability condition (11) may hold. Note also that there exist multiple attractors for $0 < \tilde{\pi} < 0.0639$. In total, we can conclude from all four panels of Figure 3 that abrupt changes in the dynamics of our model and coexisting attractors may emerge for a broad range of parameter values.

3.3. Further economic effects of nonlinear profit taxes

So far, we have seen that nonlinear profit taxes may cause abrupt changes in the dynamics of a market and lead to coexisting attractors. Before investigating the mechanisms behind these phenomena in more detail, we will show that additional exogenous shocks may increase the complexity of the dynamics of our model quite substantially, and that parameter changes may lead to hysteresis effects. Let us start with the consequences of exogenous shocks. Recall first that for our base parameter setting, the dynamics of the model is characterized by the coexistence of a low-amplitude period-two cycle and a high-amplitude period-two cycle. The top left panel of Figure 4 shows how the number of active firms evolves over time when exogenous shocks occasionally hit the dynamics (the shocks are uniformly distributed between 0 and 0.2 and arrive with a probability of 20 percent). As can be seen, the interaction between exogenous shocks, transient dynamics and attractor switching creates quite intricate fluctuations. In particular, tranquil periods alternate erratically with turbulent periods.

In the top right panel, we assume that policy-makers adjust the profit tax rate every 20 periods. In the first 20 periods, $\tau = 0.5$ and the dynamics converges towards a low-amplitude period two-cycle. Between $20 < t \leq 40$, policy-makers reduce the profit tax rate to $\tau = 0.4$, and the dynamics approaches a high-amplitude period-two cycle. Let us suppose that policy-makers are unhappy with this high-volatility regime and want to re-establish the original dynamics. Between $40 < t \leq 60$, policy-makers set the profit tax rate to the initial value, i.e. $\tau = 0.5$. As it turns out, policy-makers fail to achieve their goal: the dynamics remains on the high-amplitude period-two cycle, albeit with a slightly lower amplitude. In order to bring the dynamics back towards the initial low-amplitude period-two cycle, policy-makers have to increase the profit tax rate further. Setting $\tau = 0.54$ between $60 < t \leq 80$ significantly reduces fluctuations in the market. Once the number of firms has entered the basin of attraction of the low-amplitude period-two cycle, policy-makers can adjust the profit tax rate to its initial value $\tau = 0.5$. In fact, the dynamics between $80 < t \leq 100$ resembles the dynamics occurring between $0 < t \leq 20$.

The bottom left panel shows the mean tax revenue generated from the market when the tax rate increases from 0 to 1. As can be seen, higher tax rates tend to increase policy-makers' mean tax revenue. However, the tax revenue function is discontinuous. Increasing the tax rate above $\tau = 0.5325$ or de-

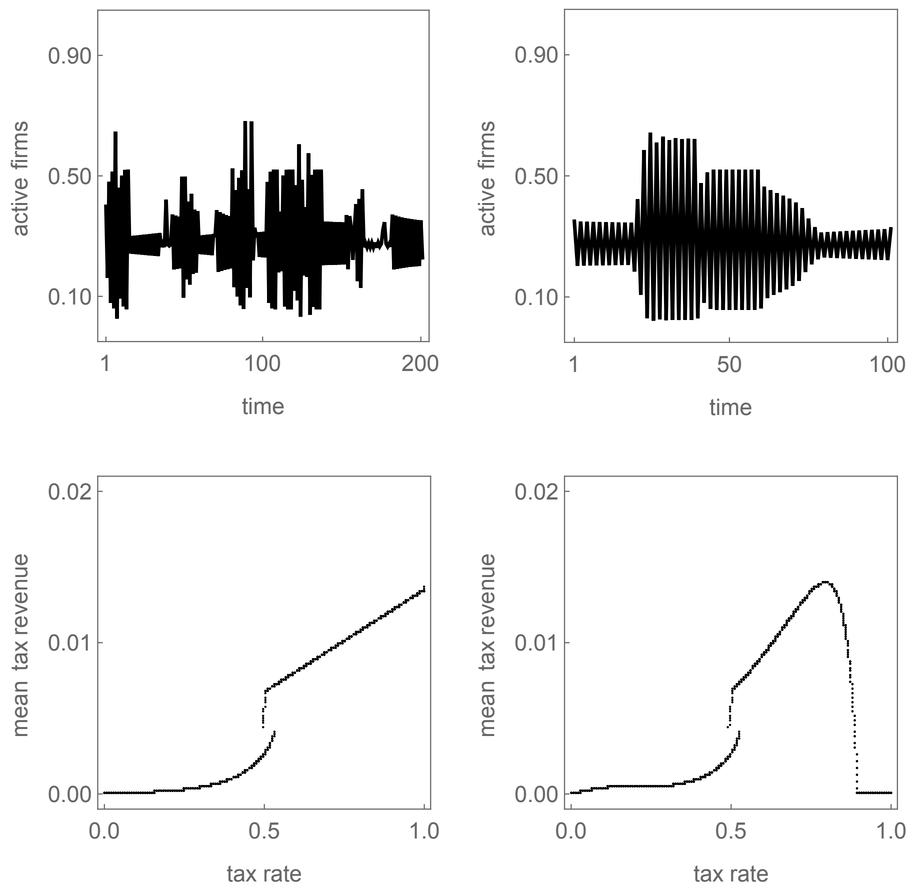

Figure 4: The top left panel shows a simulation run of map (10) with occasional exogenous shocks (the shocks are uniformly distributed between 0 and 0.2 and arrive with a probability of 20 percent). The top right panel shows a simulation run of map (10) with time-varying tax rates, i.e. $\tau = 0.5$ for $0 < t \leq 20$, $\tau = 0.4$ for $20 < t \leq 40$, $\tau = 0.5$ for $40 < t \leq 60$, $\tau = 0.54$ for $60 < t \leq 80$ and $\tau = 0.5$ for $80 < t \leq 100$. The bottom left panel shows the mean tax revenue generated from the market when the tax rate is increased from 0 to 1. The bottom right panel shows the same except that the tax rate for outside profits is fixed to 0.5, while the tax rate for the market's profits is increased from 0 to 1. Base parameter setting: $F = 0.26$, $\tilde{\pi} = 0.05$, $\tau = 0.5$ and $\beta = 42$.

creasing the tax rate below $\tau = 0.4925$ may produce significant jumps in tax revenues. Taking our simulations literally, further computations reveal that the tax revenue changes at the first discontinuity point from 0.0024 to 0.0044, i.e. by about 83 percent, and at the second discontinuity point from 0.0041 to 0.0071, i.e. by about 73 percent. Between $\tau = 0.4925$ and $\tau = 0.5325$, policy-makers' tax revenues depend on initial conditions - they can be either relatively high or relatively low. As a result, hysteresis effects may also set in. If policy-makers decrease the profit tax rate below $\tau = 0.4925$, their tax revenue may shrink substantially. This is the case when the dynamics turns from a low-amplitude period-two cycle in which firms always make a profit to a high-amplitude period-two cycle in which firms only make a profit every second period. If policy-makers want to increase their tax revenues to the old level, it may not be sufficient to return to the previous tax rate. The dynamics are most likely to remain on the high-amplitude period-two cycle on which firms' profits are lower on average. What is needed is a temporarily higher tax rate to guide the dynamics towards the less volatile original attractor.[4]

Finally, the bottom right panel of Figure 4 depicts the development of the mean tax revenue generated from the market when the tax rate for outside profits is fixed at 0.5, while the tax rate for market profits is increased from 0 to 1. For this version of our model, a typical Laffer curve emerges. For both low and high tax rates, the mean tax revenue is relatively low; for intermediate tax rates, the mean tax revenue is relatively high. The explanation for the reduction in tax revenue is that higher tax rates reduce the profitability of being active in the market and, consequently, a growing number of firms leave the market. Interestingly, this policy experiment also gives rise to hysteresis effects and a discontinuous tax revenue function. A change in the profit tax rate may cause a jump in tax revenues, and a simple return towards the original tax rate may not suffice to re-establish the previous level of tax revenues.[5]

[4] A similar picture emerges if we plot the mean tax revenue generated from the market and the firms' outside option as a function of the tax rate.

[5] We observe the same qualitative results if we set the tax rate for the firms' outside profits to lower or higher values. In fact, setting the tax rate for outside profits to zero and halving the profitability of the firms' outside option is identical to the experiment conducted above.

3.4. Appearance and disappearance of coexisting attractors

Our analysis suggests that nonlinear profit taxes may stabilize the dynamics of economic systems. Nevertheless, it is important for us to stress that such tax systems may also give rise to unexpected dynamic phenomena, an aspect that policy-makers may wish to bear in mind. Let us next explore the mechanism behind these intriguing dynamic phenomena in more detail. The two most important parameters of our model are the firms' intensity of choice and the profit tax rate. In Figures 5 and 6, we first present one-dimensional bifurcation diagrams for these parameters. In contrast to Figure 3, however, we now focus on a smaller parameter range of β and τ to highlight the area that produces multistability, that is, the area where multiple attractors coexist. In order to discuss the joint impact of the intensity of choice and the profit tax rate on the model's dynamics, we turn our attention in Figure 7 to a series of two-dimensional bifurcation diagrams. Finally, we use Figures 8 and 9 to show that the appearance and disappearance of coexisting attractors is linked directly to the tax-induced kink in the map of the model.

Let us start with the intensity of choice. The top panel of Figure 5 shows a bifurcation diagram of map (10) in which β is increased from 38.5 to 44.5. One advantage of our model is that its dynamics depends only on a single initial condition. Since the number of active firms is restricted to be between 0 and 1, we can visualize attractors' whole basins of attraction in this panel, too. As already mentioned, a fixed point $\bar{x}$ of the one-dimensional map $x_t = f(x_{t-1})$ is locally asymptotically stable if $|f'(\bar{x})| < 1$. Similarly, a period-two cycle $(\bar{x}_1, \bar{x}_2)$ of the one-dimensional map $x_t = f(x_{t-1})$ is locally asymptotically stable if $|f'(\bar{x}_1)f'(\bar{x}_2)| < 1$. In the following, we refer to such derivatives as the attractors' eigenvalues. Accordingly, the bottom panel of Figure 5 shows the evolution of the eigenvalues of attractors depicted in the top panel (the eigenvalues of the boundary steady states are outside the visible plot range). Clearly, as long as the absolute value of an attractor's eigenvalue is below one, the attractor is locally stable.

Figure 5 reveals a number of interesting insights. For $\beta < 39.63$, almost

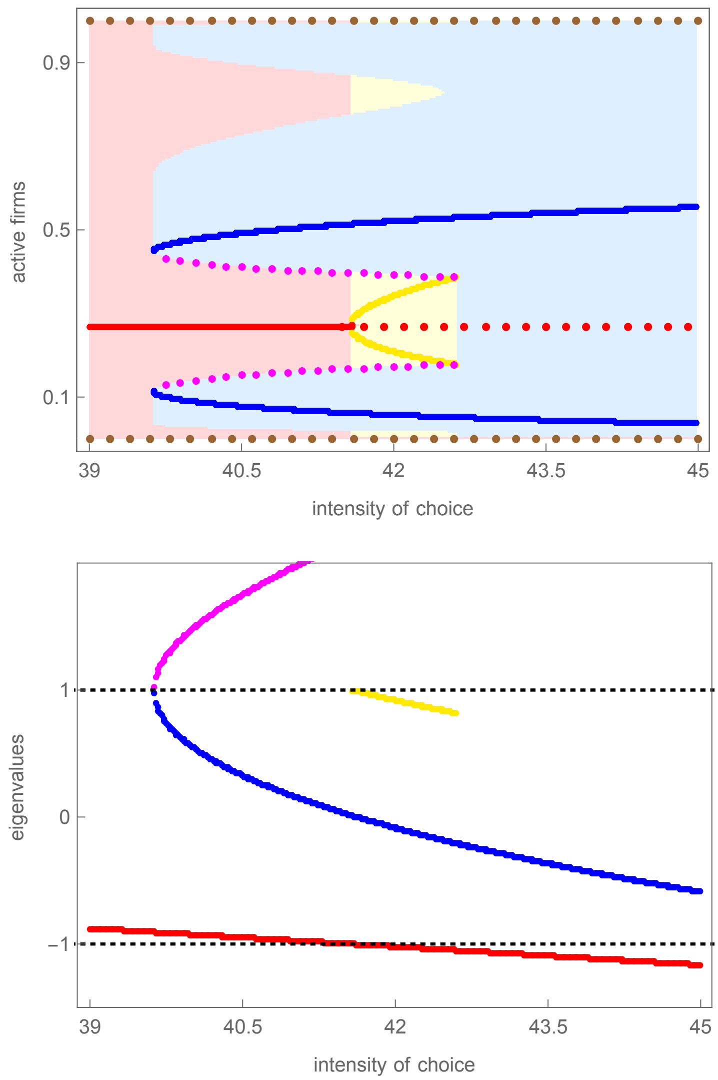

Figure 5: The top panel shows a bifurcation diagram of map (10) for the intensity of choice along with the attractors' basins of attraction. The bottom panel shows the eigenvalues of the attractors depicted in the top panel (the eigenvalues of the outer steady states are outside the visible plot range). Parameter β is increased in 200 steps from 39 to 45. Initial conditions are increased in 200 steps from 0 to 1. Other parameters: $F = 0.26$, $\tilde{\pi} = 0.05$ and $\tau = 0.5$.

all initial conditions (light red area) converge towards the inner steady state (solid red line). The only exceptions exist with the coordinates of the boundary steady states (dashed brown lines). At about $\beta = 39.63$, we observe a saddle-node bifurcation of a period-two cycle, i.e. the simultaneous birth of a locally stable high-amplitude period-two cycle (solid blue lines) and an unstable period-two cycle (dashed pink lines). Between $\beta = 39.63$ and $\beta = 41.57$, roughly half of all possible initial conditions converge to the inner steady state, while the other half (light blue area) settle down on the high-amplitude period-two cycle. The basins of attraction of the coexisting attractors - made up of several pieces - may thus be regarded as robust.

At around $\beta = 41.57$, the inner steady state becomes unstable (dashed red line) and a locally stable low-amplitude period-two cycle (solid yellow lines) emerges. Its basin of attraction consists of several parts (light yellow area). Note that these bifurcations are mirrored in the development of attractors' eigenvalues. Exactly at the period-doubling bifurcation, the eigenvalue of the inner steady state (red line) drops below -1, while the eigenvalue of the low-amplitude period-two cycle (yellow line) starts at $+1$ and then decreases. With respect to the saddle-node bifurcation of the period-two cycle, we observe that the eigenvalue of the high-amplitude period two-cycle (blue line) is $+1$ at the bifurcation and then decreases with the intensity of choice. Simultaneously, the eigenvalue of the unstable period-two cycle (pink line) is above $+1$ and increases with the intensity of choice.

At around $\beta = 42.62$, the unstable period-two cycle touches the low-amplitude period-two cycle, after which both attractors disappear. In fact, the upper of the two contact points is at $n_k = 0.387$, i.e. the number of firms at which map (10) has a kink and for which the fluctuations of the low-amplitude period-two cycle become so large that profits for active firms become negative every second period. At this point, the dynamics of the model depends on both branches of map (10). For $\beta > 42.62$, almost all initial conditions approach the high-amplitude period-two cycle. And, in fact, the only eigenvalue remaining in the stability domain is that of the high-amplitude period-two cycle.[6]

The design of Figure 6 is identical to that of Figure 5, except that we now focus on the profit tax rate. To be precise, we set $\beta = 42$ and increase the profit tax rate from 46 to 54 percent. A comparison of the top panels of Figures 5 and 6 immediately reveals that an increase in the profit tax rate reverses the bifurcation structure that emerges when the intensity of choice increases. Between $0.46 < \tau < 0.4925$, almost all initial conditions converge towards the high-amplitude period-two cycle, whereas for $0.4925 < \tau < 0.5051$, the dynamics either remains on the high-amplitude period-two cycle or approaches a low-amplitude period-two cycle. The low-amplitude period-two cycle is limited to the left by a border-collision bifurcation and to the right by a period-doubling bifurcation. At the border-collision bifurcation, fluctuations in firms' profits become so strong that the firms are about to make a loss. This is where the kink in the firms' relative profit function, respectively the kink in the model's dynamical system, comes into play. A tiny reduction in the profit tax rate increases fluctuations in the market, destroying the existence of the low-amplitude period-two cycle. At the period-doubling bifurcation, firms' market entry and exit behavior is such that the inner steady state of the model is about to become unstable. A tiny increase in the profit tax rate reduces fluctuations and dissolves the low-amplitude period-two cycle. For $0.5051 < \tau < 0.5325$, the dynamics is characterized by the coexistence of an attracting high-amplitude period-two cycle and a locally stable steady state. In order to terminate the high-amplitude period-two cycle, policy-makers have to set the profit tax rate above 0.5325. If they do this, we observe the reversal of a saddle-node bifurcation of a period-two cycle, and the market converges for almost all initial conditions towards n_2^*. The eigenvalues of attractors, depicted in the bottom panel of Figure 6, support this sequence of bifurcations.

Figure 7 depicts the joint impact of the intensity of choice and the profit

[6]Related, yet even more complex, bifurcation structures are described in Puu and Agliari (2002) and Sushko et al. (2005).

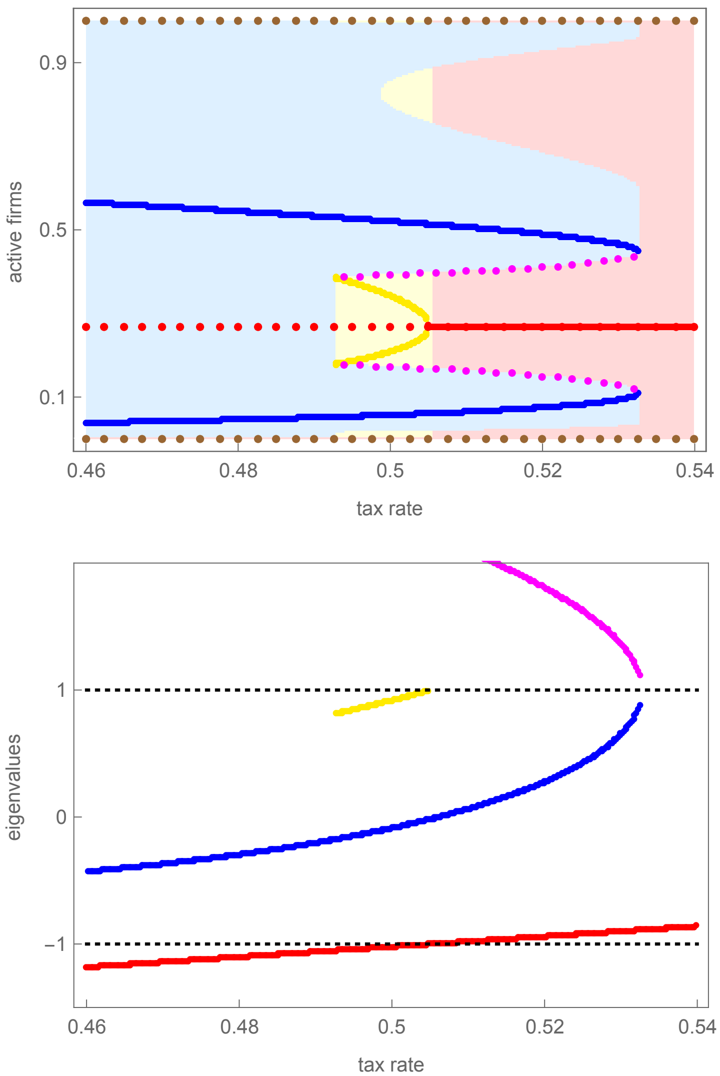

Figure 6: The top panel shows a bifurcation diagram of map (10) for the tax rate along with the attractors' basins of attraction. The bottom panel shows the eigenvalues of the attractors depicted in the top panel (the eigenvalues of the outer steady states are outside the visible plot range). Parameter τ is increased in 200 steps from 0.46 to 0.54. Initial conditions are increased in 200 steps from 0 to 1. Other parameters: $F = 0.26$, $\tilde{\pi} = 0.05$ and $\beta = 42$.

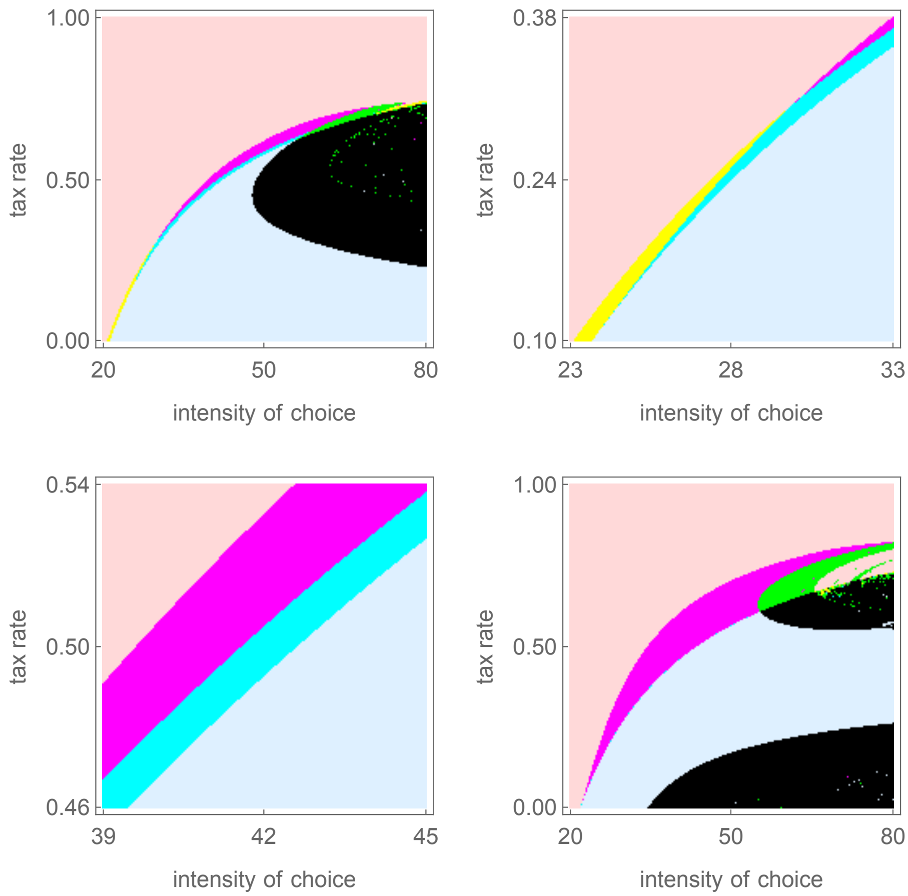

Figure 7: The four panels show two-dimensional bifurcation diagrams for the tax rate versus the intensity of choice (light red: steady state, yellow: low-amplitude period-two cycle, light blue: high-amplitude period-two cycle, magenta: steady state coexisting with high-amplitude period-two cycle, cyan: low-amplitude period-two cycle coexisting with high-amplitude period-two-cycle, green: steady state or low-amplitude period-two cycle coexisting with other attractors, black: other attractors). Parameters τ and β are varied as indicated on the axes. Simulations are based on initial conditions $n_0 = 1.005 \cdot n_2{}^*$ and $n_0 = 0.5$. Other parameters: $F = 0.26$ and $\tilde{\pi} = 0.05$, except bottom right panel in which $\tilde{\pi} = 0.01$.

tax rate on the dynamics of the model. The panels of Figure 7 are designed as follows. Parameters β and τ are varied as indicated on the axes with a grid resolution of 200 times 200. For each of these 40,000 parameter combinations, we determine the qualitative nature of the dynamics after 10,000 observations. For computational reasons, we repeat this exercise for two different initial conditions only, given by $n_0 = 1.005 \cdot n_2^*$ and $n_0 = 0.5$. The identified area of coexisting attractors thus represents a minimal area of multistability. Parameter combinations for which the dynamics always converges to the inner steady state of the model, a low-amplitude period-two cycle and a high-amplitude period-two cycle are marked light red, yellow and light blue, respectively. Parameter combinations for which the high-amplitude period-two cycle coexists with the inner steady state of the model are shown in magenta; those for which the high-amplitude period-two cycle coexists with the low-amplitude period-two cycle are marked cyan. Finally, regions shown in green stand for situations in which the inner steady state of the model or the low-amplitude period-two cycle coexist with other types of attractors and black regions capture all other types of dynamic behaviors.

In order to obtain a first picture of what could happen, in the top left panel of Figure 7 we vary the tax rate between $0 < \tau < 1$ and the intensity of choice between $20 < \beta < 80$. As already predicted by our analytical results, a higher intensity of choice requires a higher tax rate to stabilize the inner steady state of the model.[7] The top right panel of Figure 7 restricts the tax rate between $0.1 < \tau < 0.38$ and the intensity of choice between $23 < \beta < 33$. This zoom-in reveals that multistability can also be observed for lower tax rates. For instance, if the tax rate is fixed to 24 percent, an increase in the intensity of choice implies that the dynamics first converges towards the inner steady state of the model (light red), then to a low-amplitude period-two cycle (yellow), then either to a low-amplitude period-two cycle or a high-amplitude period-two cycle (cyan), and finally to a high-amplitude period-two cycle (light blue). This sequence

[7]Given $F = 0.26$ and $\tilde{\pi} = 0.05$, stability condition (11) can be expressed as $\tau = 1 - 20.7852/\beta$. For visibility reasons, we do not present this condition in Figure 7. However, the stability border of the inner steady state of the model is located where the light red region ends or, if it exists, where the magenta region ends.

of bifurcations can be observed for tax rates between approximately 13 and 30 percent. If the tax rate is below 13 percent, an increase in the intensity of choice does not lead to multistability or to abrupt changes in the dynamics (at the moment when it is born, the amplitude of the high-amplitude period-two cycle is comparable to the final amplitude of the low-amplitude period-two cycle). The explanation is that, while the map still has a kink when $\tau < 0.13$, the kink is too modest to have a significant impact on the dynamics of the model.

If the tax rate is above 30 percent, we again observe the bifurcation structure discussed in Figures 5 and 6. This can be seen better in the bottom left panel of Figure 7, in which the tax rate is increased from 0.46 to 0.54 (as in Figure 6) and the intensity of choice varies between 39 and 45 (as in Figure 5). In this parameter domain, the sequence of the saddle-node bifurcation of a period-two cycle, the period-doubling bifurcation and the border-collision bifurcation is well developed. In the bottom right panel of Figure 7, we deviate from our base parameter setting by assuming that $\tilde{\pi} = 0.01$. Obviously, for other parameter combinations the area of multistability, in particular the coexistence of the model's inner steady state with a high-amplitude period-two cycle, may be much larger and involve much smaller tax rates. In addition, even more complicated examples of multistability emerge within the green parameter area. For instance, for $\tau = 0.65$ and $\beta = 62$, the inner steady state of the model is locally stable and coexists with a chaotic attractor.

Figures 8 and 9 demonstrate how the shape of the model's map and its dynamic properties depend on the intensity of choice and on the profit tax rate. The aim of these figures is to show that the appearance and disappearance of coexisting attractors is caused directly by the tax-induced kink in the map. The left panels of Figures 8 and 9 present the first iterate of map (10), that is $n_{t+1} = f(n_t)$. These panels also contain a number of typical trajectories, for which the color coding corresponds to that used in Figures 5 and 6. As already mentioned, the steady states of our model are located where map (10) crosses the 45-degree line. Steady states and period-two cycles of our model can be

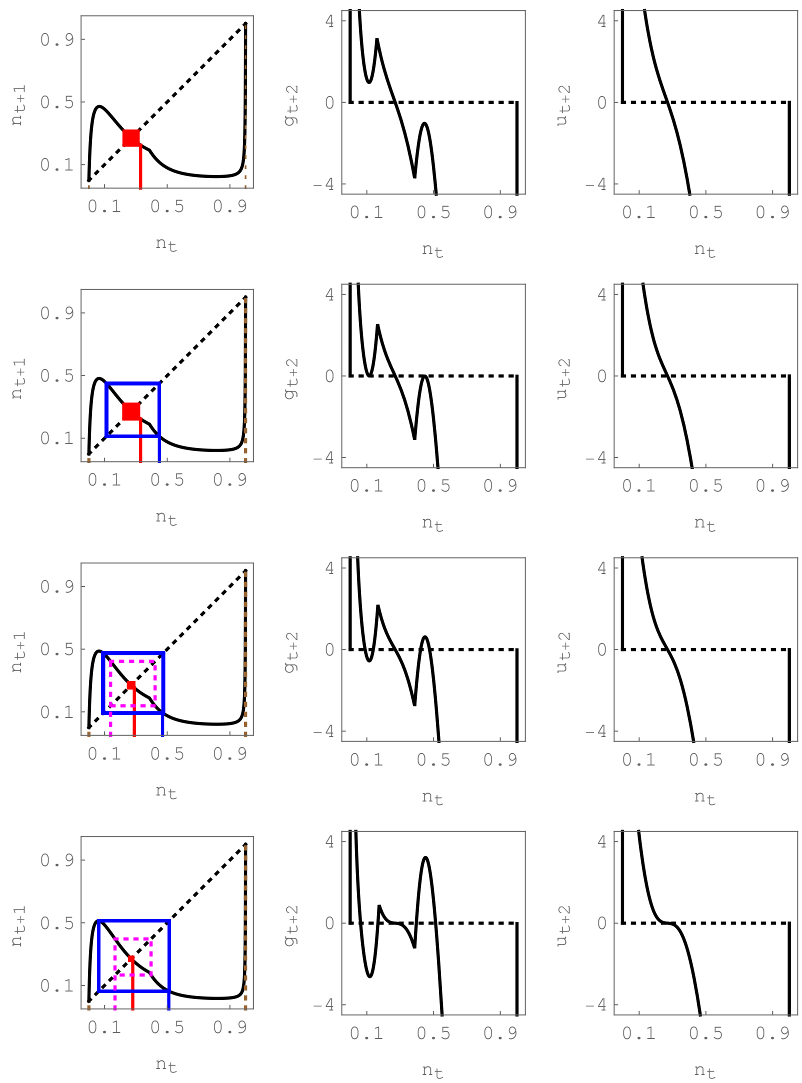

Figure 8: The first and transformed second iterate of map (10) and the transformed second iterate of the upper branch of map (10) for different values of the intensity of choice, respectively. The left panels also contain some exemplary trajectories. First line: $\beta = 39$. Second line: $\beta = 39.63$. Third line: $\beta = 40$. Fourth line: $\beta = 41.5$. Other parameters: $F = 0.26$, $\tilde{\pi} = 0.05$ and $\tau = 0.5$.

found where the second iterate of map (10), that is $n_{t+2} = f(f(n_t))$, crosses the 45-degree line. For reasons of clarity, the central panels of Figures 8 and

9 show the function $g_{t+2} = 100(f(f(n_t)) - n_t)$, which we call the transformed second iterate of map (10). Note that g_{t+2} indicates the change in the number of firms active between period t and period $t+2$. For $g_{t+2} > 0$, the number of active firms increases over the next two periods, while for $g_{t+2} < 0$ it decreases. Accordingly, steady states and period-two cycles of our model are located where the transformed second iterate of map (10) becomes zero. The right panels of Figures 8 and 9 show the transformed second iterate of the upper branch of map (10), which we denote by u_{t+2} for convenience. Restricting the dynamics to the upper branch of map (10) means economically that firms receive a tax compensation if they make a loss and technically that such a model version is free of any kinks. Comparing functions g_{t+2} and u_{t+2} thus enables us to identify the effects of the tax-induced kink on the dynamics of the model.[8]

In the first line of panels in Figure 8, we set $\beta = 39$. As a result, the inner steady state is locally stable; the boundary steady states are unstable; and there are no period-two cycles. We already know from Figure 5 that for this parameter constellation almost all initial conditions approach the inner steady state. One example of a trajectory is plotted in the left panel. The transformed second iterate of map (10) is zero at the three steady states of the model. Note that the same is true for u_{t+2}. In the second line of panels in Figure 8, we set $\beta = 39.63$. Now the transformed second iterate of map (10) additionally touches the zero line twice, implying the birth of a high-amplitude period-two cycle. Accordingly, we see in the left panel two trajectories, one of which converges to the inner steady state while the other converges to the high-amplitude period-two cycle. From the moment the period-two cycle starts

[8] An alternative experiment to check the effects of such a tax-induced kink would be to set the tax rate to zero and to halve the intensity of choice. As long as the dynamics of this alternative experiment remains within the upper branch of map (10), it would be identical to that discussed above. Without a profit tax, however, there is no difference between the upper and lower branch of map (10), and therefore this alterative experiment does not have a kink either. Both experiments produce very similar results. In fact, for the parameter settings used in Figures 8 and 9, virtually no graphical differences are visible between the two experiments.

to exist, it is characterized by a high amplitude and, as we know from Figure 5, roughly approached from half of all possible initial conditions.

Even if the analysis turns out to be cumbersome, a good understanding of the causes of the model's bifurcation structure necessitates a deeper discussion of the properties of g_{t+2}. First of all, g_{t+2} has two kinks. The kink at $n_t = 0.387$ is the one we already know from map (10). Note that g_{t+2} changes direction at this kink and possesses a local maximum for a slightly higher value of n_t. Since g_{t+2} decreases between the model's inner steady state and $n_t = 0.387$, an increase in n_t implies that the number of active firms will decrease more strongly in this region over the next two periods (which pushes the dynamics towards n_2^*). At $n_t = 0.387$, the number of active firms is such that their profits are zero. If n_t exceeds 0.387, firms suffer a loss and the kink in the firms' profit function comes into play. Suppose that firms (start to) make losses in period t. Since the number of active firms sharply reduces in period $t + 1$, profitability spikes simultaneously. As a result, relatively more firms become active again in period $t + 2$. In the first line of panels in Figure 8, the local maximum of g_{t+2} is still negative, implying that more firms in total will exit the market over the next two time steps. In the second line of panels in Figure 8, the local maximum of g_{t+2} is exactly zero. Now the number of active firms will not change over the next two time steps, i.e. a high-amplitude period-two cycle is established. At the upper value of this cycle, firms make substantial losses and thus retreat from the market. At the lower value of this cycle, firms make substantial profits and re-enter the market. Both forces are balanced exactly at the high-amplitude period-two cycle.

A similar interpretation can be given for the kink at $n_t = 0.1605$ (which is, since $f(0.1605) = 0.387$, the preimage of the kink at $n_t = 0.387$). Transgressing $n_t = 0.1605$ from right to left reduces the increase in the number of active firms within the next two periods. The reason for this is as follows. In the case of a low number of active firms, profitability is so high that the inflow of additional firms is such that profits become negative in the next time step. As a result, many firms retreat from the market. For $\beta = 39.63$ and $\tau = 0.5$, the retreat of firms may exactly offset the initial inflow of firms, which completes the high-amplitude period-two cycle described above. It is this complicated bending of g_{t+2} that first creates and, as we will see in the sequel, eventually destroys the phenomenon of coexisting attractors. In contrast, the right panels in Figures 8 and 9 reveal that such kink-induced bending is absent for u_{t+2}.

In the third line of panels in Figure 8, we set $\beta = 40$. As can be seen, the transformed second iterate of map (10) crosses the zero line seven times and, consequently, the map is characterized by a locally stable inner steady state; two unstable boundary steady states; a locally stable high-amplitude period-two cycle; and an unstable period-two cycle.[9] From the path of g_{t+2} it also becomes clear that the unstable period-two cycle separates the basin of attraction of the model's inner steady state from that of the high-amplitude period-two cycle (see also the dotted pink line in Figure 5). For instance, g_{t+2} is negative (positive) to the left (right) of the upper value of the unstable period-two cycle, implying that the number of active firms is pushed towards the inner steady state of the model (upper value of the high-amplitude period-two cycle). In the fourth line of panels in Figure 8, we set $\beta = 41.5$. The inner steady state is about to lose its local stability, the fluctuations of the high-amplitude period-two cycle have increased while the amplitude of the unstable steady state has decreased. This is the situation just before the analytically detected primary bifurcation of the inner steady state.

Figure 9 continues the analysis of Figure 8. In the first line of panels in Figure 9, we set $\beta = 42$. For this parameter constellation, the transformed second iterate of map (10) is zero at nine positions. In total, there are three unstable steady states; an unstable period-two cycle; a locally stable high-amplitude period-two cycle; and a locally stable low-amplitude period-two cycle (see also Figure 1, which is based on the same (base) parameter setting). Note that

[9]The computation of the unstable period-two cycle, as depicted in Figures 5, 6, 8 and 9, rests on finding the intersection points of the transformed second iterate of map (10) with the zero line via a grid-search procedure.

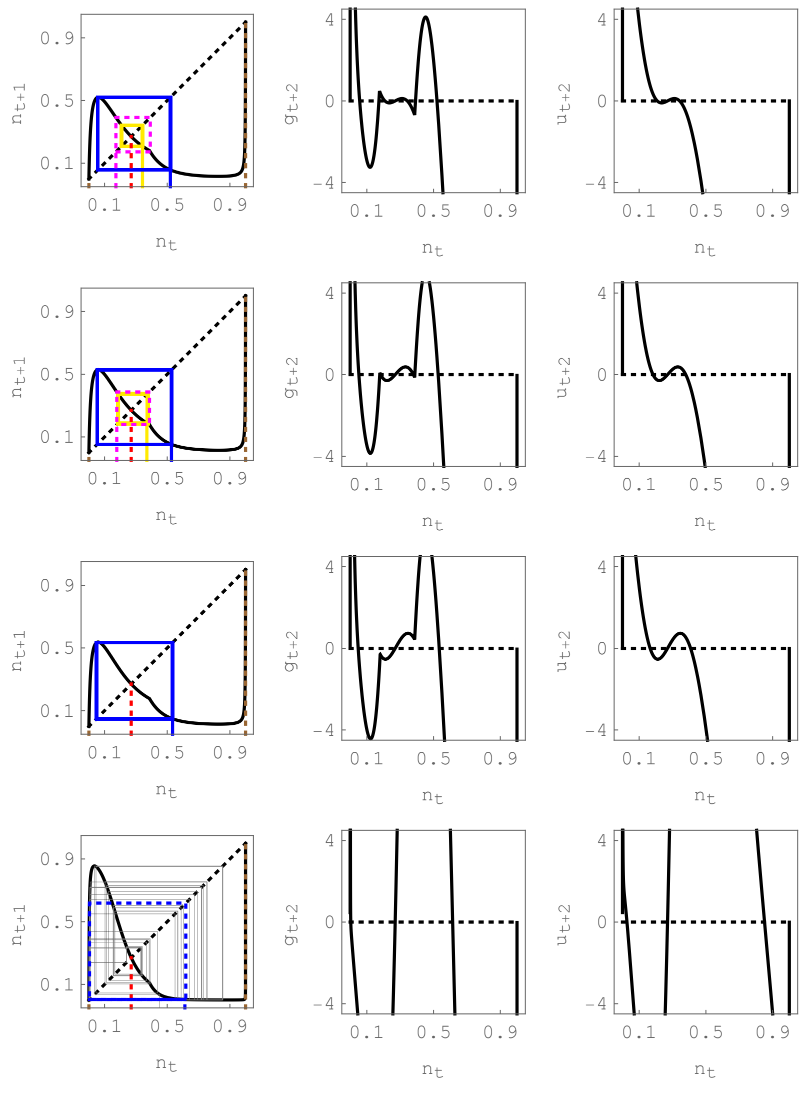

Figure 9: The first and transformed second iterate of map (10) and the transformed second iterate of the upper branch of map (10) for different values of the intensity of choice, respectively. The left panels also contain some exemplary trajectories. First line: $\beta = 42$. Second line: $\beta = 42.5$. Third line: $\beta = 43$. Fourth line: $\beta = 65$. Other parameters: $F = 0.26$, $\tilde{\pi} = 0.05$ and $\tau = 0.5$.

g_{t+2} and u_{t+2} exhibit the same bending near the inner steady state of the model. Since the slope of g_{t+2} (and of u_{t+2}) is positive at n_2^*, the model's inner steady state is unstable. The economic reason for this is that firms react quite aggressively to profit differentials. At the low-amplitude period-two cycle, the market entry and exit behavior of firms is balanced. Since the market's profitability is positive along this cycle, we also observe it for u_{t+2}.

In the second line of panels in Figure 9, we set $\beta = 42.5$. Note that the unstable steady state and the low-amplitude period-two cycle have become very close and are about to touch. In the third line of panels in Figure 9, we set $\beta = 43$. Since the low-amplitude period-two cycle and the unstable period-two cycle vanish after they touch, the transformed second iterate of map (10) crosses the zero line only five times. Now almost all trajectories converge to the high-amplitude period-two cycle. This is a dramatic change in the model dynamics. For slightly lower values of the intensity of choice, the dynamics could have settled on a low-amplitude period-two cycle. Comparing g_{t+2} and u_{t+2} reveals again that this is due to the tax-induced kink of the model which implies that g_{t+2} changes its direction at the kink, destroying the low-amplitude period-two cycle. Clearly, u_{t+2} still indicates the existence of an attracting low-amplitude period-two cycle for the kink-free alternative model. In the fourth line of panels in Figure 9, we set $\beta = 65$. While there still exist three unstable steady states and an unstable high-amplitude period-two cycle, the dynamics is now chaotic (the bifurcation diagram in the top left panel of Figure 3 reveals that chaos emerges for a broad range of β values). Further simulations (not depicted) indicate that the alternative model is still characterized by an attracting period-two cycle (this cycle also remains the unique attracting cycle for much higher values of the intensity of choice).

4. Robustness checks

In order to demonstrate that our main results do not hinge on the specific setup of our stylized behavioral market entry model, we discuss a number of robustness checks in this section. In particular, we modify consumers' demand, vary firms' expectation formation and market entry behavior, and endogenize the firms' outside option. Our results, summarized in Figure 10, indicate that the main bifurcation sequence discussed in this paper may be regarded as an astonishingly robust phenomenon. Despite substantial model changes, effecting the equations of the model, its dimension and parameters, we can detect abrupt changes in the dynamics of the model, coexisting attractors and hysteresis effects.

In the top left panel of Figure 10, we have substituted the linear demand function (2) by the isoelastic demand function $D_t = \frac{k}{p_t}$ with $k = 0.25$. Parameter β serves as our bifurcation parameter; the other parameters are $F = 0.2$, $\tilde{\pi} = 0.05$ and $\tau = 0.5$. Note that we have here, amongst other things, an example where the low-amplitude period-two cycle coexists with a high-amplitude period-four cycle. In the top right panel of Figure 10, we assume that firms have naive expectations $p_t^e = p_{t-1}$ instead of rational expectations (5). Such a modification turns our model into a two-dimensional map. Nevertheless, coexisting attractors and abrupt changes in the dynamics of our model can be observed. The downward shift of attractors after the saddle-node bifurcation of a period-two cycle reflects a drop in firms' profits due to their use of a naive expectation rule.

In the center left panel of Figure 10, we exchange the exponential replicator dynamics (9) by the discrete choice approach $n_t = \frac{exp[\beta \pi_{t-1}]}{exp[\beta \pi_{t-1}] + exp[\beta \hat{\pi}]}$. The discrete choice approach is very popular within heterogeneous agent models, see, e.g. the seminal contributions by Brock and Hommes (1997, 1998). In the center right panel of Figure 10, we also use the discrete choice approach, but additionally assume that firms rely on naive expectations. In such versions of our model, the inner steady state depends on the intensity of choice; it is not guaranteed that the market's profitability at the inner steady state is equal to the profitability of the firms' outside option. From this perspective, the discrete choice approach has stronger behavioral implications for the learning behavior of firms. In any case, the bifurcation structure of the original model survives these model changes (the parameter setting for the left and right panel is $F = 0.2$,

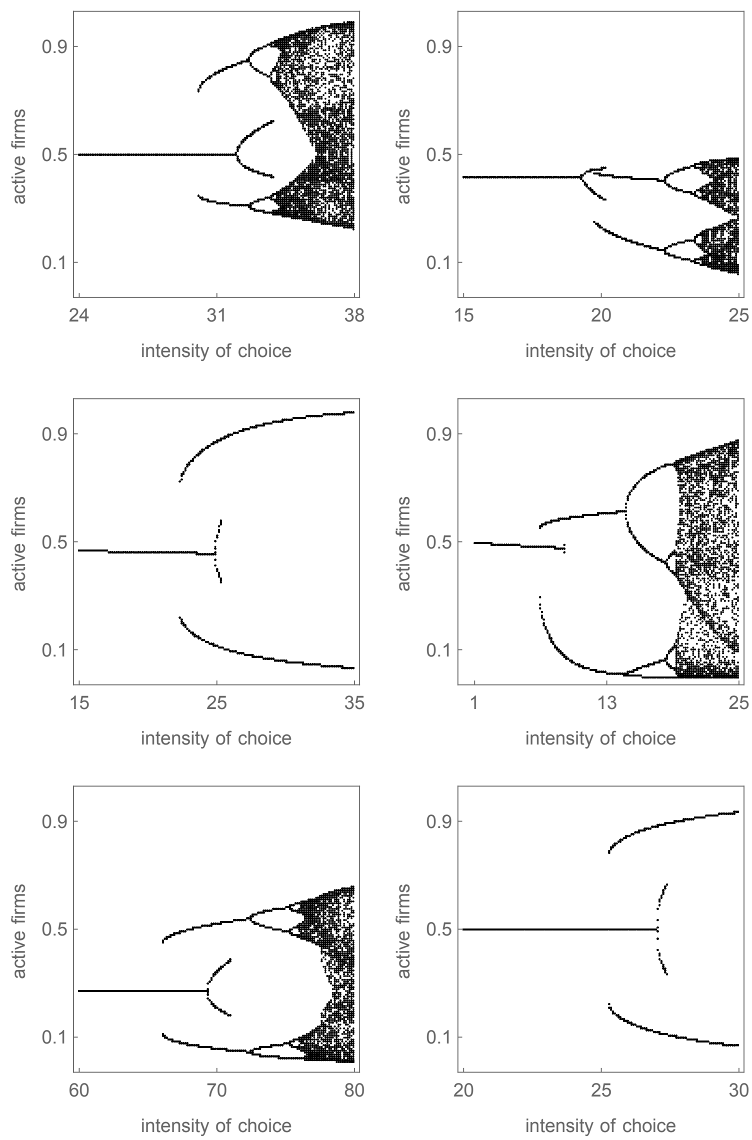

Figure 10: Bifurcation diagrams for different model versions with respect to the firm's intensity of choice. Top left: Isoelastic demand function. Top right: Naive expectations. Central left: Discrete choice approach. Central right: Discrete choice approach and naive expectations. Bottom left: Memory in the fitness function. Bottom right: Endogenous outside profits. Parameter settings are reported in Section 4.

$\tilde{\pi} = 0.05$ and $\tau = 0.5$).[10]

In the bottom left panel of Figure 10, the firms' market entry behavior depends on a smoothed profit measure, i.e. $v_t = (1 - m)\pi_t + mv_{t-1}$, where the firms' memory is bounded between $0 < m < 1$. Accordingly, the replicator dynamics (9) turns into $n_t = \frac{n_{t-1}exp[\beta v_{t-1}]}{n_{t-1}exp[\beta v_{t-1}]+(1-n_{t-1})exp[\beta\tilde{\pi}]}$. Results are displayed for a memory of $m = 0.25$, while the other parameters are $F = 0.26$, $\tilde{\pi} = 0.05$, and $\tau = 0.5$. Although this adaptation increases the dimension of our model again, the emerging bifurcation pattern is very close to what we observed for our original model. Similar results may also arise for much higher memory parameters (for $m = 0.75$, for instance, the model's inner steady state is abruptly accompanied by a high-amplitude period-three cycle). In the bottom right panel of Figure 10, the firms' outside option, guaranteeing constant exogenous profits, is endogenized by another market. For simplicity, we assume that markets are symmetric. As it turns out, this model remains one-dimensional, has one less parameter and, since profits in both markets can become negative, depends on two (symmetric) kinks.[11] For $F = 0.18$ and $\tau = 0.5$, we recover our standard bifurcation structure.

5. Conclusions

There may be many reasons for introducing nonlinear profit taxes. For instance, our stylized behavioral market entry model predicts that higher profit tax rates may be beneficial for market stability. The main reason for this outcome is that higher profit tax rates reduce fitness differentials between firms' strategies and, consequently, slow down their market entry and exit behavior.

[10] Schmitt and Westerhoff (2015) find that a piecewise-linear profit tax may also trigger abrupt changes in the dynamics and multistability within the original cobweb model of Brock and Hommes (1997). However, that model setup precludes a deeper and clear-cut analysis of the phenomena discussed in this paper.

[11] In the future, we plan to explore such a two-market model in more detail. For instance, an interesting question is whether policy-makers are able to stabilize asymmetric markets by imposing market-specific tax rates.

With less drastic changes in aggregate supply, the market displays less extreme price fluctuations. However, our analysis also reveals that a piecewise-linear profit tax creates a kink in the firms' profit function which, in turn, causes a kink in the dynamical system of the market. The effects of such a tax-induced kink may be troublesome:

- First of all, we find that abrupt changes in the dynamics of a market may emerge when a low-amplitude period-two cycle turns into a high-amplitude period-two cycle via a slight change in a model parameter (border-collision bifurcation). Abrupt changes in the dynamics of a market may also emerge when a high-amplitude period-two cycle turns into a steady state via a slight change in a model parameter (saddle-node bifurcation of a period-two cycle). Of course, both bifurcation scenarios imply a dramatic shift in the dynamic behavior of the market. In particular, the border-collision bifurcation gives rise to a spontaneous jump in the volatility of market prices and exchanged quantities.

- Between these two bifurcation phenomena, there may be a robust parameter range in which a locally stable steady state or a low-amplitude period-two cycle coexist with a high-amplitude period-two cycle. If such an environment is subject to noise, the combination of exogenous shocks, transient dynamics and attractor switching may lead to very intricate dynamics and occasional volatility outbursts. This observation may also help us to explain the complex behavior of many markets. In reality, volatile markets are typically regarded as harmful.

- The bifurcation structure of the model may furthermore give rise to hysteresis effects. Suppose, for instance, that the market's dynamics is characterized by a low-amplitude period-two cycle that coexists with a high-amplitude period-two cycle. A reduction in the profit tax rate may destroy the low-amplitude period-two cycle and push the dynamics to a high-amplitude period-two cycle. Unfortunately, a return to the previous profit tax rate may not suffice to drive the dynamics back to the low-amplitude period-two cycle since the system may remain in the basin of attraction of the coexisting high-amplitude period-two cycle. A temporary increase to a higher profit tax rate may then be needed to guide the dynamics back towards the low-amplitude period-two cycle.

- Such hysteresis effects may also be relevant for policy-makers' tax revenues. Even very small changes in the profit tax rate may cause substantial jumps in tax revenues, leading to either much higher or much lower tax revenues. Due to the coexistence of attractors, a return to the previous profit tax rate does not necessarily mean that the previous tax revenue can be realized again. Of course, a high-amplitude period-two cycle implies - at least in contrast to a steady state or a low-amplitude period-two cycle - that policy-makers' tax revenues are also subject to stronger fluctuations.

These effects are quite robust and can, among others, also be observed in a model with an isoelastic demand function, naive expectations, memory in the firms' fitness function or an endogenous outside market. Note that these alternative model building blocks do not only change the functional form of the model's dynamical system, but also its dimension. Put differently, our results do not hinge on the assumption that the dynamics of our model is driven by a one-dimensional map.

Although we have already extended our model in various directions, more could be done in the future. In our current setup, policy-makers' tax revenues do not re-enter the economy. One could assume that a certain proportion of the tax revenue goes to firms, lowering their effective fixed costs, and to consumers, increasing their total demand. It would also be interesting to turn our partial equilibrium model into a general equilibrium model and to study how piecewise-linear tax measures, producing the peculiar bifurcation structure discussed in this paper, affect the welfare of firms and consumers.

Since we are unaware of any empirical evidence that can be used to test the predictions of our model, it would be interesting to test our model in a laboratory setting. Do piecewise-linear profit taxes have a stabilizing effect? Does a piecewise-linear profit tax give rise to multistability? In this respect, it is worth mentioning that Agliari et al. (2005) are able to explain the experimental evidence of Hommes et al. (2005) according to which the same asset-pricing experiments can lead either to fixed point dynamics or to strong oscillatory motion. Agliari et al. (2015) explain this fact in their nonlinear asset-pricing model via the coexistence of attractors. Depending on initial conditions, the dynamics may settle on a calm or turbulent attractor. Clearly, path dependence is also an issue in our model.

To sum up, policy-makers can stabilize the dynamics of our model by imposing profit taxes. As long as the stability-ensuring profit tax rate is rather low, an increase in the tax rate can lead to a smooth stabilization of the dynamics. However, when the stability-ensuring tax rate is sufficiently high, an increase in the tax rate may cause a number of unintended phenomena. An important take-away message of our paper is that a piecewise-linear profit tax causes a kink in the firms' profit function (and thus in the dynamical system of the market) and that this kink may cause a substantial change in the firms' behavior. We believe that this observation is not only relevant in our model, but may also have serious consequences for many other market environments in which agents face such a type of regulation and base their behavior on past profits.

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